NATURE'S GEOMETRY

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1. Introduction

Geometry essentially deals with two types of entities: objects and spaces. Points, circles, lines, curves, cylinders, tetrahedrons — these are examples of the *objects* one finds in geometry books. These objects live in *spaces* that hold them. A point (a zero-dimensional geometrical object) can rest on a one-dimensional line or a two-dimensional plane. A curve (a one- dimensional geometrical object) can rest on a 2-D surface or in a 3-D volume. We thus have objects embedded in spaces of dimension greater than or equal to that of the objects.

For much of human history, geometers majorly studied idealized objects with regular shapes: triangles, circles, spheres and rectangular parallelopipeds. This was the legacy of Pythagoras and Euclid. The embedding space was always thought to be Euclidean: a flat piece of paper in case of 2-D space, a volume with 'flat' characteristics in case of 3-D space etc.

Towards the end of the 19th century, there was a revolution in geometry. Lobachevsky, Riemann and others showed that space can also be curved, and then Einstein found a profound application of the new geometry in his theory of gravitation. At least in the domain of spaces the narrow confines of Euclidean space was broken.

But objects still remained Euclidean. We still talked about angles contained in *triangles* in Euclidean space and non-Euclidean space. The objects of study in geometry continued to be idealized ones that one can think up in one's head but never finds in nature.

Nature abounds in irregular objects. "Mountains are not cones", as Benoit Mandelbrot, the founder of the new geometry put it, "clouds are not spheres, lightnings are not straight lines". Towards the end of the twentieth century we seem to be breaking out of the compartment of Euclidean objects. Geometers are now considering these irregular objects as valid subjects of study. And that is what fractal geometry is all about.

2. Euclidean objects versus natural objects

What distinguishes natural objects from Euclidean objects? The first thing that comes to one's mind is the irregularity of the shape of natural objects. But that needs to be specified in mathematical terms.

If we take a very small length of a curve, say $y = f(x)$, it approximates a straight line. The closer we look at it, the more it loses its structure. That is how we can define the first derivative as a limiting value of the rate of change. Same is the character of Euclidean surfaces—they smoothen out into flat planes as you take a close look.[1]

Here lies the main distinction between natural objects and the idealized objects. Natural objects never flatten out—at what-

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[1]The derivative is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This limit would exist only if the curve smoothens out into the tangent as $\Delta x \to 0$

ever level of magnification you may look at them. Think of any natural surface — the surface of your skin or that of a tree trunk. They contain complexities within complexities which come in view at higher levels of magnification. These are continuous but nondifferentiable surfaces.

This applies to some data sets also. No economist ever tries to differentiate the curve for share prices or exchange rates. No electrical engineer would think of differentiating the curve representing the load on a power station. One cannot take the derivative of the curve for oscillation during earthquakes. The reason is, these curves as geometrical objects, are continuous but not differentiable anywhere. They reveal more and more rich forms as you zoom closer.

The next major difference is illustrated by Mandelbrot's famous question, "how long is the coastline of England?". A little thought makes it evident that the measured length of the coastline depends on the yardstick of measurement. If the yardstick is long, much of the detail of coastline geometry would be missed. As smaller and smaller yardsticks are used, the creeks and bends come in view and the measured length increases. And in the limit—when the yardstick length shrinks close to zero—the length of the coastline becomes infinitely large.

Yet the area of England *is* finite. Thus the coastline is a curve of infinite length enclosing a finite area.

Same is the case of all 3-D natural objects enclosed by natural surfaces. Take the structure of our lungs as an example. The task of the lungs is to absorb oxygen from air. In order to have a high absorption rate, the surface area needs to be very high. Yet, the volume must be small—it has to be accomodated within the rib cage. Thus it has the same geometrical characteristics as the coastline. You can multiply examples just by looking around yourself.

These objects, evidently, need a new mathematical tool for characterization. There comes the question of *dimension*.

3. Dimension of geometrical objects

It is strange that in school and college level mathematics curricula we never learn what 'dimension' means in geometry. Our conception, naturally, is mostly derived from common sense: if we have a line-like object we say it is one dimensional, if we have a surface-like object we say it is two dimensional and so on.

We are accustomed to thinking of dimensions as integers, and have considerable difficulty in visualizing anything otherwise. The ancients had the same sort of difficulty when the only numbers they knew were natural numbers. Three horses, ten men, twentifive bananas — such numbers came naturally to them. But finally they had to shake off the narrow confines of integers and conceive fractions — in order to represent length, area, weight etc.

It now appears that we again have to shake off the notion of integral dimensions and conceive fractional dimensions when we face the task of representing natural objects.

It must first be understood that the dimension of any object and that of the embedding space are two different quantities. The dimension of the embedding space is given by the degree of freedom. In 1-D space one can move only left or right, in 2-D space one can move left-right as well as front-back, in 3-D space one can also move up-down. The embedding dimension, naturally, has to be an integer.

The dimension of an object can not be defined in a similar manner. It has to be defined according to the way *it fills space.* To probe the question, let us take a simple Eu-

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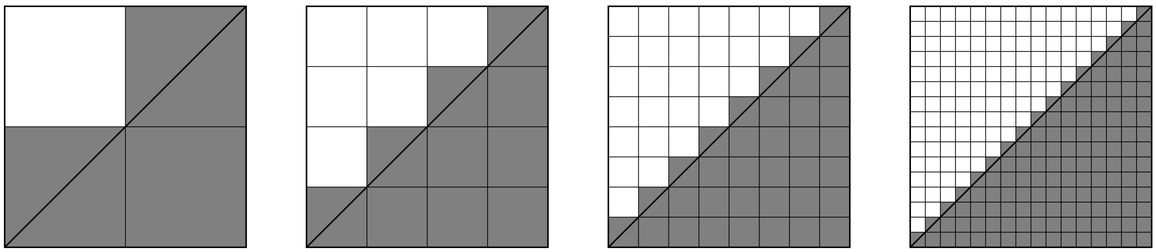

Figure 1: To find how a triangle fills space, it is covered by a grid and the grid size is successively reduced. The boxes required to cover the object is shown in shade. In the limit we find that the triangle fills space in the same way as the square. But the boundary of the carbon particle seen in the electron microscope photograph does it in a different way.

clidean object: the square. We know that it is two dimensional. But how do we obtain the number 2 from the structure of this object?

In order to see how it fills space, we cover it with a grid as shown in Fig.1. If the square has 1 cm sides and the distance between two consecutive grid lines is 1/10 cm, then 100 grid elements would be necessary to cover the square. If we now reduce the grid length by half, 400 grid elements would be required. We see that the number of grid elements required to cover the object increases as the square of the reciprocal of grid length.

$$N(\epsilon) = \left(\frac{1}{\epsilon}\right)^2$$

where ϵ is the grid length and $N(\epsilon)$ is the number of grid elements required to cover the object (a function of ϵ).

If the object taken is a right angled triangle, the count of covering boxes finally converges to

$$N(\epsilon) = \frac{1}{2}\left(\frac{1}{\epsilon}\right)^2$$

And for a circle we have

$$N(\epsilon) = \frac{\pi}{4}\left(\frac{1}{\epsilon}\right)^2$$

To generalize, we can write

$$N(\epsilon) = K\left(\frac{1}{\epsilon}\right)^2$$

where K is a constant.

We can extract the dimension (2 in this case) from it as follows:

$$\ln N(\epsilon) = \ln K + 2\ln\frac{1}{\epsilon}$$

$$2 = \frac{\ln N(\epsilon)}{\ln\frac{1}{\epsilon}} - \frac{\ln K}{\ln\frac{1}{\epsilon}}$$

The second term would vanish as $\epsilon \to 0$. Thus the dimension D of the object is given by

$$D = \lim_{\epsilon\to 0}\frac{\ln N(\epsilon)}{\ln\frac{1}{\epsilon}}$$

If instead of the known Euclidean objects we take some natural object like a carbon particle or a map of the Andamans, and subject it to the above procedure, we would find that the dimension turns out to be a fraction. This is a general characteristics of all natural objects as distinct from idealized objects. The fractional dimension thus provides a method of characterizing natural objects. In fact, such objects are *defined* by this property.

Geometrical objects with fractional dimensions are called *Fractals*.

4. What use is fractal dimension?

How do we quantify the fractal character of an object? What experimental procedure would enable us to judge if two dissimilar fractal objects are closely related or geometrically equivalent?

Looking at any object, we always have a intuitive feeling about how densely the object occupies space, how crooked, twisted, broken it is. Looking at a curve plotted from a set of data, we do feel how wavy or "noisy" it is. But these are subjective feelings. We need a concrete objective methodology to assess this quality. Measurement of the fractal dimension provides the means to achieve this end.

The method can easily be guessed from the definition of dimension given in the last section. We divide the embedding space into a number of equal "boxes". In case of 2-D space, we divide the sheet of paper into small squares as in a graph paper. In case of 3-D space we divide it into cubes.

Then we count how many of these elemental boxes are required to cover the object. This is our N. The side of the box is ϵ. Subsequently, reduce the size of the box in steps and repeat the procedure of counting. Then plot D versus ϵ. The plot approximates a horizontal straight line as ϵ tends to zero. When a reasonable approximation is obtained, the point where it cuts the y-axis gives the dimension of the object (Fig.2).

But what use in this new piece of information? It actually quantifies the surface characteristics. And if any of the properties of the body is determined by the characteristics of the surface, fractal dimension can supply vital information.

For example, the fractal dimension of the surface of the carbon particles in automobile exhaust is related to the afficacy with which the particle will attach with the breathing ducts and cause harm.

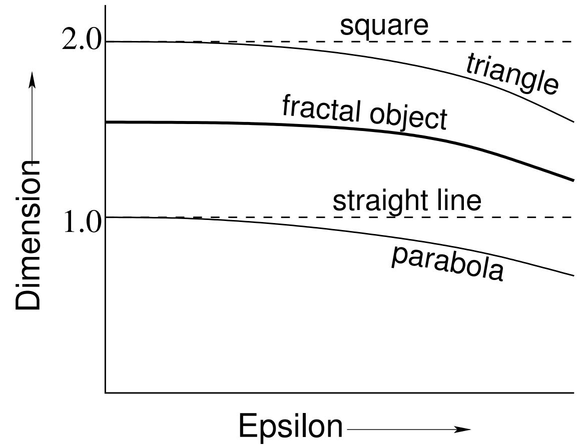

Figure 2: Measurement of the fractal dimension of a natural object.

When two surfaces make electrical contact (as in a switch), all the points never touch each other. The amount of actual electrical contact is determined by the fractal character of the surfaces, and that is quantified by the dimension of the two surfaces.

Some silt particles float in river water and some settle quickly. The precipitation obviously depends on the size and specific gravity of the particles. But these two parameters being equal, the more crooked the surface, the more it will get carried by flowing water. This property is again quantified by the fractal dimension.

When metals, semiconductors or alloys crystallize, the crystal grains have boundaries that are fractals. Scientists have found a number of properties of such materials that are related to the fractal character of the grains.

Any surface is fractal. However much you may polish and smoothen a surface, at some level of magnification irregularities must show up. And the light absorption property of the surface is determined not only by the property of the material but also by the character of the surface. This, again, can be quantified by fractal dimension.

When rock structures form in various geological processes, the characteristic signature of the formative process is left in the geometry of the rocks. And these are nothing but fractals. You may look at large mountain ranges, large boulders, small pebbles or miniscule grains that become visible when you venture inside the rocks by cutting them — you see structures inside structures — that wait to tell you much about their character.

The afficacy of solid catalysts in helping or retarding a reaction depends on how much surface area is exposed to the reactants. Same is the case for the electrodes in electrochemical reactions. None of these surfaces are smooth, and it is easy to see how their behaviour would be related to the fractal dimension of the surfaces in question.

The veins in the plant leaves, similarly, are fractals. The arteries and veins in your body are fractals. Tumours and cancer cells are fractals. The economists' data set for inflation rates are fractals. And in all these cases, the fractal dimension gives a direct measurement of the character of the object.

5. Mandelbrot and Julia sets

Can we generate geometrical objects with such characters mathematically? Benoit Mandelbrot considered this problem first and came up with a solution. He found that such structures can not be generated by the known mathematical procedures of writing equations and functions. For it, one has to follow the procedure of repeating the same operation again and again.

Suppose there is a system whose status is changing continuously. In scientists' perlance it would be called a dynamical system, and it is easy to appreciate that all natural systems are dynamical systems. Scientists try to express the changes or "dynamics" by equations so that the status at any point can be computed from its history.

Generally the dynamical equations may be quite complicated. But simplified versions often reveal properties observed in complicated systems as well. Let us take a simple dynamical system expressed by the equation:

$$Z_{t+1} = (Z_t)^2 + C$$

where Z_t is the state of the system at the t-th instant and Z_{t+1} is the state at the next instant. The Z's and C are complex numbers which have a "real part" and an "imaginary part" expressed as $a + ib$; and i is the "imaginary" number $\sqrt{-1}$. Such complex numbers can be plotted as points on a "complex plane" with the real part along the x-axis and the imaginary part along the y-axis. Dynamics of the complex number Z can then be viewed as the changing position of a point on a sheet of paper — the complex plane.

If we choose a value of C and a starting point Z_0, we can calculate subsequent values of Z and observe the dynamics. We find that for some values of Z_0 the system remains bounded, for some other values Z increases without bounds, i.e., runs towards infinity. If we now plot those values of Z_0 for which the system remains within bounds, we get a set of points making up a picture. This turns out to be a fractal—the Julia Set (Fig.3).

We can also vary C while taking the starting point Z_0 same in all computations. We find that for some values of the parameter C the system remains within bounds and for some other values it doesn't. We can again plot on a sheet of paper the locations of those values of C which make the system bounded. We again get a fractal—the Mandelbrot set (Fig.4).

The Julia set and the Mandelbrot set have become sort of a universal representation of

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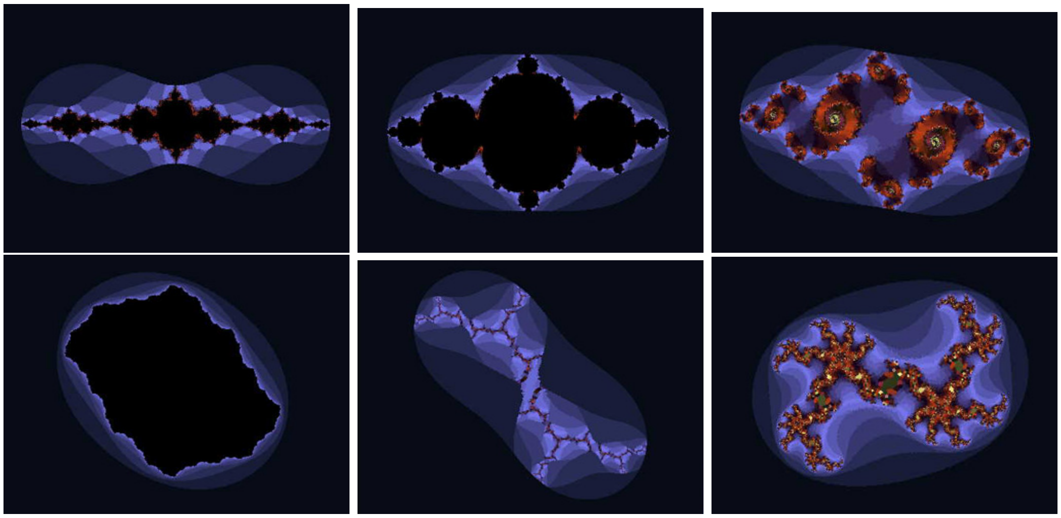

Figure 3: Examples of Julia Sets

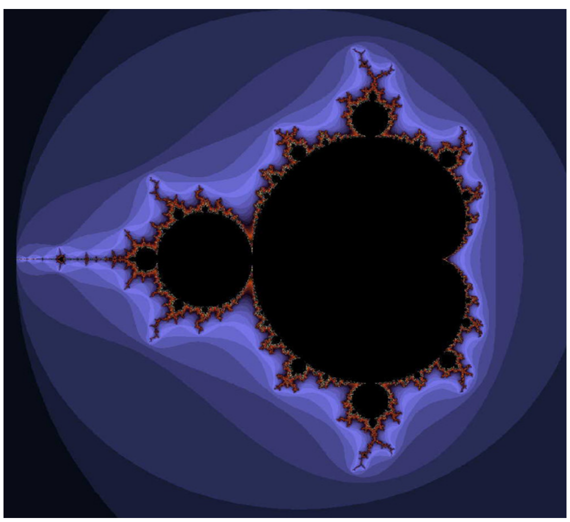

Figure 4: The Mandelbrot set. In this geometrical object, however small part of the figure you may want to observe, it reveals rich internal structures. The same is true for Julia sets and all other fractal objects.

fractal geometry. Most books, articles and popular expositions on fractal geometry begin with them and end with them, emphasizing their symmetry, beauty, complexity and all that, obtained from the simple equation $Z_{t+1} = Z_t^2 + C$. And in most cases the basic idea is lost: they are derived from a simplified representation of a natural process, a dynamical system.

Nature is full of dynamical systems. And for each one there would be certain *parameters* and certain *initial conditions*. The lesson that we learn from Mandelbrot's work is that for every dynamical system there exist fractals in the space formed by the system parameters and in the space formed by the initial conditions. Most dynamical systems in nature, in fact, work on fractal geometry.

Let us take an example from engineering. A ship is stable in the upright position and tends to come back to this position if slightly disturbed from it. But with excessive amount of tilt, it may capsize. Thus, depending on the initial condition of the tilt, its state may be bounded (upright position) or can escape without bounds (capsize). A real ship in an ocean would continuously be bombarded by waves, and its dynamics would depend on the intensity of the waves (let it be denoted by F). Now we write down the equation of motion of the system[2], and find out which initial conditions keep the system bounded. We can now plot these initial conditions in a plane with the tilt angle in the x-axis and the rate of change of tilt angle in the y-axis. We get a picture. If we now change the value of F in steps and draw the picture again, we find that it changes dramatically with increasing wave intensity. The boundary of the zone for stable initial conditions can not be properly discerned and the initial conditions from which the system collapses get mixed up with it. The picture becomes a fractal (Fig.5). While rocking in the wave, if the state of the ship ever touches any of the white dots, it will capsize. Thus fractal structures are of vital importance to engineers. And it is generated by the same methodology as Julia sets.

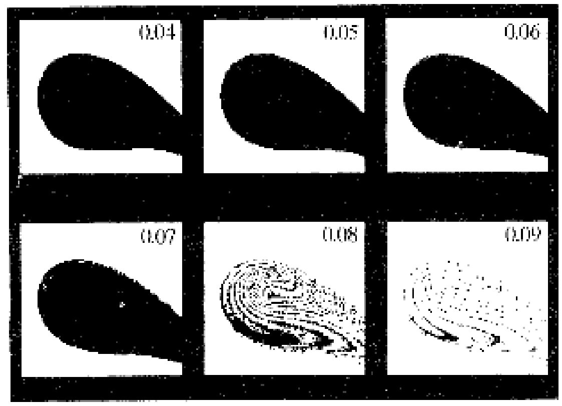

Figure 5: The set of initial conditions for which a ship remains stable becomes fractal under certain conditions. This is the reason for many cases of ship capsize.

6. Iterative Function Systems

There is an entirely different approach to the problem of modelling real-life objects with mathematical methods. Developed by Professor Michael Bernsley of Georgia Institute of Technology, USA, this method uses elegant mathematical reasoning to generate images of natural objects on the computer screen.

Any black and white two-dimensional picture is nothing but a set of black dots in a white background. It can be viewed as a *set of points* in a 2-D space. Now, any point in the 2-D space can be located with the help of two real numbers: the first number giving the distance that we go along the x-axis and the second number for the distance we go parallel to the y-axis in order to reach the point. Hence a sheet of paper, in the view of mathematics, is a *space* formed by two real numbers. Any picture drawn on a paper is nothing but a set of points in that space, that is, a collection of *pairs* of two real numbers.

How can you reach a point starting from another point? Here comes the concept of *functions*. We know the functions of a real number, like $f(x) = a.x + b$. Here, x is a real number and and $f(x)$ is also a real number. So both the starting point and the destination are elements of the real line (Fig.6). The function (or mapping) carries a point on the real line to another.

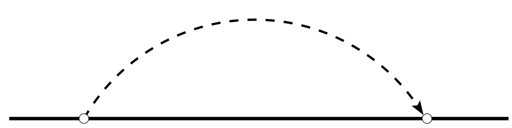

Figure 6: A function takes one point on the real line to another.

There can be such functions in the 2-D space also — carrying a point on a sheet of

[2]Neglecting unnecessary details, the equation may be written as

$$\ddot{x} + \beta\dot{x} + x - x^2 = F \sin \omega t$$

where β represents the frictional damping and waves of intensity F strike the ship with a frequency ω. For the computations that generated the pictures, we took $\beta = 0.1$ and $\omega = 0.85$

paper to another. Such a simple function of the form

$$x_2 = ax_1 + by_1 + e$$

$$y_2 = cx_1 + dy_1 + f$$

is called affine transformation.

We can apply such transformations on *all* the points of a figure. Thus, while we can get one point from another by application of the transformation, we can also get a whole geometrical figure from another using the same transformation. And by application of the transformation again and again we get a "sequence" of figures.

The question is, will such a sequence lead us anywhere? We know that a *convergent* sequence of numbers (like $1, \frac{1}{2}, \frac{1}{4}, ...$) always has a specific number as the limit point, and for any given number we can always define a suitable converging sequence to give that number. If the sequence of figures is made convergent, it will also have a particular figure as its limit. This is in fact guaranteed by a theorem called "Banach fixed point theorem".

In order to get such a meaningful sequence we only need to ensure that the sequence of figures is convergent. That is ensured if two points obtained from the transformation are closer to each other than the original pair of points. Such transformations are called contraction mappings.

Thus we can get any figure by suitably defining a contraction mapping. For complicated constructions, we only have to define a number of such transformations, each giving a figure. The resultant figure would simply be the combination (or union) of these sub-figures.

In this method it is immaterial from which figure we start. We can as well start from a square. On repeated application of the transformations (iterations) the figure will gradually change shape before your eyes and "converge" on to the figure you want. The collection of contraction mappings that generate a particular picture is called the "iterative function system". The picture of the fern shown in Fig.7 was generated from a square, by iterating a function system comprising four affine transformations defined by a,b,c,d,e and f as follows.

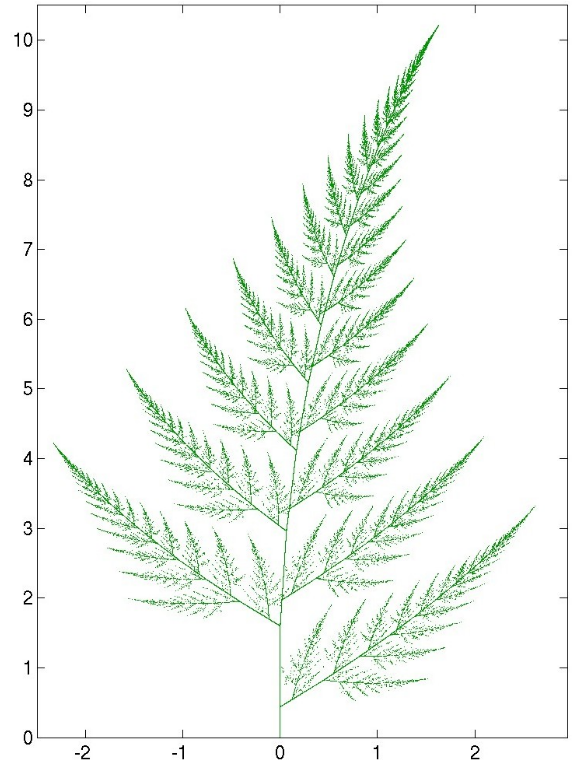

Figure 7: Successive application of the function system transforms a square into a fern. All information about this immensely complicated picture is contained in just 24 numbers. For any figure, it is possible to define similar iterated function systems.

fn	a	b	c	d	e	f
1	0	0	0	0.16	0	0
2	0.85	0.04	-0.04	0.85	0	1.6
3	0.2	-0.26	0.23	0.22	0	1.6
4	-0.15	0.28	0.26	0.24	0	0.44

It has been shown that *all* geometrical figures that can be drawn on a piece of paper can be generated from such transformations. We only have to find the transformations for any particular figure. The mathematics to do this has also been developed. We can thus squeeze the information contained in pictures and photographs in the from of just a few numbers.

This has immense technological consequence. For example, when spacecrafts far in space send photographs, the picture is divided into grids and the grey level in each grid has to be coded and transmitted to earth. For good pictures this needs long transmission time as a huge amount of information has to be sent. Now the fractal image compression offers the possibility of compressing the image into a few numbers enabling very fast transmission. Programs for coding the image into such numbers has been developed and may soon become an international protocol for satellite transmission.

7. Last Words

Much of the popular literature on fractals present it as a mathematical game of generating fancy pictures in a computer. In contrast, this article presents it as a necessary tool to model the real geometry of nature. In fact in recent years the discovery of fractal geometry has caused a sea change in the geometers' approach. We are no longer thinking up "perfect" mathematical shapes and specifying apriori what nature must be like. We are now taking real lessons from nature.

This has led to new ways of measuring the complexity of natural objects. And we find that the fractal dimension quantifies diverse phenomena in nature, which have real scientific and technological significance.

In an attempt to model the natural objects scientists have discovered that very simple rules, applied in an iterative way, can generate extremely complicated structures. Does it hold a clue to the immense complexity of natural objects? It has always intrigued scientists how the information stored in a single molecule—the DNA—can generate an unimaginably complex entity like a human body. Now we know it is possible, mathematically.◇